Bode’s Law

Estimates of the distances between the first six planets and the sun were known in the late 18th century. It was obvious that the distances between successive planets increased as one moved further from the sun, and it no doubt seemed logical to pose that some natural law should be proposed for these spacings.

Someone got the idea of writing down some regular, simple sequences that approximated the planetary distances, and eventually references to them in writing were made by Johann Titius and Johann Bode. Although neither had discovered these sequences, both their names were associated with them. Bode mentioned them so often in his writings that his name stuck, while Titius tended to be forgotten except by historians of astronomy. Probably the first algebraic expression of Bode’s Law was by Johann Wurm in 1787:

\[ d = 0.387 + 0.293 \cdot 2^{n-2} \]

where \( d \) is the orbital distance, with the earth’s distance taken as 1, and \( n \) is the sequential number of the planet, starting with \( n=1 \) for Mercury.

**Checking the law.** The history of Bode’s law descended into murkiness almost immediately from its first appearance. There was substantial argument about who had discovered it, who had copied it, and from whom. Bode himself virtually never compared his version of the law with the known planetary distances, a lack of curiosity that seems strange until one remembers that in his era many scientists were better at investing faith than making observations.

We can get an idea of how well Wurm’s formula works in the next graphic, which needs a little explanation. First, it was realized from the very first that the fifth prediction did not match the fifth planet (Jupiter). But if Jupiter were moved to sixth position, then the accuracy of the predictions greatly improved. This was explained at the time by saying that there must be a planet between Mars and Jupiter, which would surely be discovered. I will explain the fifth planet in the graphic momentarily. Secondly, the planets were only known out to Saturn (planet 8 in the graphic), and for them the fit of Bode’s law seemed quite good. Bode’s fortunes rose appreciably in 1781 when Neptune (planet 9) was discovered, and its distance fit reasonably well with Bode’s law. Even more fortunately, Ceres (planet 5 in the graphic) was discovered in 1801 and it fit remarkably well. Bode was exceedingly pleased with these discoveries. He was
less pleased a few years later with another “planet” of the size of Ceres was discovered near it, and presumably if he had lived long enough he would have become even more dismayed at the regular discovery of new asteroids.

Fit of Bode’s Law to our solar system (including Ceres as planet 5)

One of the of the problems with the graphic is that it represents values that are on different scales. This means that some features will be emphasized while others are suppressed. We can fix this by taking logs (natural logs, ln, actually) of the distances. This is shown in the next graphic, which also reveals that Mercury is not well fitted. This was recognized in Bode’s time.
In summary, Bode’s law was originally cobbled together by someone (before Titius and Bode) who was perhaps looking for some numerological significance in the placement of the planets. Although Bode’s law has been examined in modern times to see whether it follows from some dynamic theory of how the solar system came to be, this was completely impossible for late 18\textsuperscript{th} century astronomers. The fact that Wurm’s version did a poor job for Mercury was kind of ignored, which seemed justified when the law was confirmed by the discoveries of Uranus and Ceres. It is worth remembering that Wurm’s formula has three parameters, and at the time it was only claimed to fit 5 planets, along with one more that did not exist at the time.

The checkered history of the announcement of the law, along with its sketchy fit to the data and complete lack of theory, caused it to be regarded with derision by most professional astronomers. This was picked up by science dabblers, who were pleased to inform us that it was at best an example of clumsy pre-science, or maybe just pseudo-science.

While there is much to criticize about Bode’s Law, its fit to the data cannot really be disparaged. The relatively poor fits for Mercury and Pluto (discovered
Bode’s Law

in 1930) is typical; models often fit the data less well at the extremes than in the center, and Bode does very well in the center. Moreover, we now know that there is a considerable amount of mass in the asteroids between Mars and Jupiter, where the missing planet should be. It would be interesting to compute a mass-weighted average distance of the asteroids to see whether that would fit Bode’s Law, perhaps indicating a plant that came apart or somehow failed to form (which was also suggested in the early 19th century as more “minor planets” were discovered). And of course relatively recently Pluto was downgraded from “planet” to “minor planet” that might be seen as taking it out of the scope of Bode’s Law. Stacked against modern scientific data analyses Bode’s Law has better data fit and more plausible supporting arguments than many.

**Log-linear Modelling.** Modern analysts who want to discuss Bode’s Law have used the logs of the distances, based presumably on something like the last graphic. The model equation is

\[ \ln(d) = \alpha + \beta \cdot n \quad \text{or} \quad d = (e^\alpha)(e^\beta)^n \quad (d = \text{distance}, \ n = \text{sequence}) \]

The estimate of \( e^\alpha \) is 0.218 and for \( e^\beta \) it is 1.710. (An empty orbit is put at \( n=5 \).) The next graphic gives the support functions, and the model fit is shown in the graphic after that, indicating that it is even better than Bode’s Law.

Support functions for the two values determining the log-linear model fit to our solar system (Ceres excluded)
Log-linear fit to the planets of our solar system (Ceres excluded)

It is always worthwhile to look at the residuals from a model; that is, the observed values minus the fitted values. This sometimes provides important additional information about how the sample was actually produced. The next graphic shows residuals on the log scale, with the remarkable suggestion of a sinusoid. Because sinusoids tend to be solutions of dynamic differential equations, it is conceivable that this might play a role in some theory of creation of the solar system.
Bode’s Law

Residuals in the log-linear fit to our solar system.

Validation by replication. Bode’s Law is sometimes portrayed as one of those potentially interesting facts that is not of any actual scientific importance. One reason is that there is only one solar system, so no replication is possible. This was certainly true in Bode’s day, but the argument is now out of date. In the next three graphics we can see fittings of log linear models to the major satellites of Jupiter, Saturn, and Uranus. The fits for Jupiter and Uranus are good, but there are only four or five satellites, and we are fitting with two parameters. Saturn looks less convincing, but this may not be definitive because Saturn also has a considerable number of substantial rings, which (like the asteroid belt) may represent understandable departures from Bode’s law.
Bode’s Law

Log-linear fit to the moons of Jupiter.

Log-linear fit to the moons of Saturn.
Log-linear fit to the moons of Uranus.

This is not the limit of possible replications. We now have data on satellites from star systems like our solar system. In these systems the planets are referred to as “exoplanets”. The following three graphics show results for three systems that were selected to be the best, the middle, and worst from a larger series (Altaie 2016).
Bode’s Law

Log-linear fit to an exoplanetary system.

Log-linear fit to an exoplanetary system.
**Progression of spacings.** If you read about the earliest proposals and discussions of Bode’s Law, you will find that it came in several versions. The basic idea of each was, however, that the spacings between planets formed a geometric progression; each was twice the previous one. In the log-linear model, subsequence spacings follow

\[
\frac{d_{n+1} - d_n}{d_n - d_{n-1}} = \exp(\alpha + \beta(n + 1)) - \exp(\alpha + \beta n) = e^\beta
\]

Thus \(e^\beta\) is free of the units used to measure distance, and 2 was the original idea. Perhaps the slightly different versions were attempts to make this true. In particular, one can add a constant to term to the log-linear form, recovering Wurm’s form of the law, and the above equation remains true. The good log-linear fits suggest, however, that the constant is unnecessary. In any case we can see that \(e^\beta\) does indeed capture the assumption that the spacings are in a progression, and here are the values for the models I fitted above:
Whether these should be considered estimates of some universal constant can be investigated by looking at a graph of their individual support functions, shown below. Recall that the support for two values being equal is the value of the support where their support functions cross. So while there may be some cases where equality is supported, and these might be worth further study, for now we can say that there is certainly heterogeneity, and so probably not any universal constant, and even if there were it would not be 2.

<table>
<thead>
<tr>
<th>System</th>
<th>$e^\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solar*</td>
<td>1.71</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1.64</td>
</tr>
<tr>
<td>Saturn</td>
<td>1.64</td>
</tr>
<tr>
<td>Uranus</td>
<td>1.47</td>
</tr>
<tr>
<td>Kepler-444</td>
<td>1.18</td>
</tr>
<tr>
<td>Kepler-215</td>
<td>1.56</td>
</tr>
<tr>
<td>HD 160691*</td>
<td>1.78</td>
</tr>
</tbody>
</table>

*sequence numbers modified

**Support functions for the estimates of $e^\beta$ from the solar and exoplanetary systems.**

**Inventing empty orbits.** We have seen in two cases that the imputation of empty orbits was required to get a good fit. We can revisit Saturn in this regard, by trying the injection of various empty orbits, the example graphed
Bode’s Law

below showing three injections. Many scientists would say that by artfully inventing empty orbits any data could be made to fit Bode’s Law. This would mean that there is no way to falsify it, so it is not really part of science. But the issue is not as clear as it might seem. If we had an experiment that could be re-run, then arbitrarily modifying observations would certainly be suspect. But all of the datasets we have are for the most part one-time observations. The scientific issue with Bode’s Law is whether there is some dynamic process that can describe satellite formation which would give log-linear orbital distances. If there is, then manipulating sequence numbers is not much different from trying different parameter values to fit the model. The astronomical observations have been obtained at such a high price, and are so unique, that it seems almost disrespectful to fail to try to squeeze as much meaning from them as possible. And, of course, in the case of exoplanetary systems we may eventually find occupants for the empty orbits.

\[
\begin{align*}
\text{ln} (\text{Distance}) & \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \\
12 & 13 & 14 & 15 & 16 \\
\end{align*}
\]

Saturn (modified)

Log-linear fit to Saturn’s moon, with empty orbits posited to improve the fit.

**In summary.** Bode’s Law has a number of features that seem to recur in the history of science. The original idea had an actual basis, but it was formulated or estimated in a clumsy or inaccurate matter. There was no obvious
meaning to regularity in the spacings of satellites, and certainly no theory. Opinions then split, with some scientists thinking it was nonsense and others (later on, in the modern period) revisiting it to see whether it made more sense in the light of the additional background information that had been acquired over time. As we have seen, if we drop the intercept term in Bode’s Law, obtaining a log-linear model, the values that summarize the original idea of proportional spacings turn out to be modestly consistent, given the wildly different circumstances from which they were obtained, but each system has its own progression of spacings.

Textbooks and conventional histories frequently hide these aspects of science. They prefer to show science as an inexorable march of progress, with one success building on previous successes. They want to throw away the milk and save the cream. This is, however, an essentially false story about how the human activity of science evolves. It is mostly error and frustration, with occasional epiphanies, and actual scientists put up with the annoyances to get the epiphanies.

Resources
Addendum: More Complete Moons of Jupiter

Since finishing the material above, I found more complete data on Jupiter’s moons from NASA. The log-linear fit was

\[ \ln(\text{orbit}) = 4.283 + 0.450 \times \text{sequence} \]

and the plot of \( \ln(\text{distance}) \) against serial number is

![Plot of ln distance against sequence number, for the seven known inner moons of Jupiter.](image-url)
The support function for $e^\beta$ is

Support function for the Bode parameter in the case of Jupiter’s seven inner moons.

There is an additional sequence of moons, but the nearest is seven times the last one in this sequence, suggesting they do not follow the same law.